GORENSTEIN FANO THREEFOLDS WITH BASE POINTS IN THE ANTICANONICAL SYSTEM

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1. Introduction

In the classification of Fano varieties, those which are not "Gino Fano", i.e., for which $-K_X$ is ample but not very ample, are usually annoying. In the beginning of his classification of Fano threefolds Iskovskikh listed those for which $|-K_X|$ is not free. The purpose of this article is to see how his result extends to the canonical Gorenstein case.

If X is a Gorenstein Fano threefold with at worst canonical singularities, and $\mathrm{Bs}\,|-K_X| \neq \emptyset$, then the rational map defined by $|-K_X|$ goes to a surface W, which is a rational ruled surface Σ_e with $e \geq 0$ or $\widehat{C}_d$, the cone over a rational normal curve of degree d. The following Theorem lists the possible pairs (X, W):

1.1. Theorem. *Let X be a Gorenstein Fano threefold with at worst canonical singularities and $\mathrm{Bs}\,|-K_X| \neq \emptyset$. Then we are in one of the following cases.*

i) $\dim \mathrm{Bs}\,|-K_X| = 0$. *In this case X is a complete intersection in $\mathbb{P}(1^4, 2, 3)$ of a quadric Q, defined in the first four linear variables, and a sextic F_6; $(-K_X)^3 = 2$ and W is the quadric Q in $\mathbb{P}_3$.*

ii) $\dim \mathrm{Bs}\,|-K_X| = 1$. *Then $\mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1$ and either*
   (a) *X is the blowup of a sextic in $\mathbb{P}(1^3, 2, 3)$ along a complete intersection curve of arithmetic genus 1; $(-K_X)^3 = 4$ and $W \simeq \Sigma_1$ or*
   (b) *$X \simeq S_1 \times \mathbb{P}_1$, where S_1 is a del Pezzo surface of degree 1 with at worst Du Val singularities; $(-K_X)^3 = 6$ and $W \simeq \mathbb{P}_1 \times \mathbb{P}_1$ or*
   (c) *$X = X_{2m-2}$ is an anticanonical model of the blowup of the variety U_m (see below) along a smooth, rational complete intersection curve $\Gamma_0 \subset U_{m,\mathrm{reg}}$ for $3 \leq m \leq 12$; $(-K_X)^3 = 2m - 2$ and $W \simeq \widehat{C}_m$.*

Here U_m denotes a double cover of $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(m) \oplus \mathcal{O}_{\mathbb{P}_1}(m-4) \oplus \mathcal{O}_{\mathbb{P}_1})$ with at worst canonical singularities, such that $-K_{U_m}$ is the pullback of the tautological line bundle $\mathcal{O}(1)$. For $m \geq 4$, this is a hyperelliptic Gorenstein almost Fano threefold of degree $4m - 8$. The curve Γ_0 lies over the complete intersection of some general element in $|\mathcal{O}(1)|$ and the "minimal surface" $B \in |\mathcal{O}(1) - mF|$, where $|F|$ denotes the pencil (note that Γ_0 is always contained in the ramification locus). If $m = 3$, then Γ_0 is the only curve, on which $-K_{U_3}$ is not nef. For details of the construction see section 5.

The cases (a) and (b) are as in Iskovskikh's list. In a different context case i) appears in [Me99] and [IT01], and apparently also in [M88].

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2. Preliminaries

We recall the following fundamental results:

2.1. Theorem [Shokurov, [Sho80]**/Reid,** [R83]**].** *Let X be a Gorenstein Fano threefold with at worst canonical singularities. Then $|-K_X|$ contains an irreducible surface S with at worst Du Val singularities, called* general elephant.

The birational contraction $h\colon Y \to X$ in the following theorem is called a *partial crepant resolution* or *terminal modification* of X:

2.2. Theorem [Reid, [R79]**/Kawamata,** [K88]**].** *Let X be a threefold with only canonical singularities. Then there exists a $\mathbb{Q}$–factorial threefold Y with only terminal singularities and a birational contraction $h\colon Y \to X$ such that $K_Y = h^*K_X$.*

If X is Gorenstein, then Y is in fact factorial (for example [K88], Lemma 5.1.).

A Gorenstein threefold X for which $-K_X$ is big and nef is called *almost Fano.* It is called *hyperelliptic*, if $|-K_X|$ is free, but the associated map φ fails to be injective at the generic point. In that case

$$\varphi\colon X \longrightarrow W \subset \mathbb{P}_N$$

is generically 2–to–1 and W is a so–called variety of minimal degree, i.e.,

$$\deg W = \mathrm{codim} W + 1.$$

Varieties of minimal degree have been classified by del Pezzo ([dP85]) in dimension 2 and by Bertini in arbitrary dimension n ([Ber07]). The list (with some repetitions) is as follows:

i) $\mathbb{P}_n$;
ii) the n–dimensional quadric $Q_n \subset \mathbb{P}_{n+1}$;
iii) (a cone over) the Veronese surface;
iv) (a cone over) a rational scroll.

The *cone over a (rational) scroll*, denoted $\overline{\mathbb{F}(d_1,\dots,d_n)}$, is the image of

$$\mathbb{F}(d_1,\dots,d_n) = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_1}(d_n)), \quad d_1 \geq \cdots \geq d_n \geq 0$$

in $\mathbb{P}_{d_1+\cdots+d_n+n-1}$ under the map associated to the tautological line bundle which will be denoted $\mathcal{O}(1)$. Note that for $d_n \geq 1$, $\overline{\mathbb{F}(d_1,\dots,d_n)}$ and $\mathbb{F}(d_1,\dots,d_n)$ are isomorphic. The pencil on $\mathbb{F}(d_1,\dots,d_n)$ will be denoted by $|F|$.

Any effective divisor D on $\mathbb{F}(d_1,\dots,d_n)$ is in a system

$$D \in |\mathcal{O}(k) - lF|, \quad k \geq 0 \text{ and } l \in \mathbb{Z}.$$

Fiberwise, $D \cap F$ is a hypersurface of degree k in $\mathbb{P}_{n-1}$. If $x_1,\dots,x_n$ denote homogeneous coordinates of $\mathbb{P}_{n-1}$ corresponding to the summands of our vector bundle, then the monomial $x_1^{e_1}\cdots x_n^{e_n}$ with $e_1+\cdots+e_n = k$ has as coefficient a function taken from

$$H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(e_1d_1+\cdots+e_nd_n - l)).$$

We will use this in the following form. Consider $\mathbb{F}(m, m-4) \simeq \Sigma_4$. Denote by ξ_4 the minimal section. Any divisor

$$D \in |\mathcal{O}(k) - lF|, \quad k \geq 0 \text{ and } l > k(m-4)$$

contains ξ_4 as a component. Indeed, using the above notation, ξ_4 corresponds fiberwise to $x_1 = 0$. It therefore suffices to prove that the coefficient function of x_2^k vanishes. This is a section of $\mathcal{O}_{\mathbb{P}_1}(k(m-4) - l)$, so the claim follows.

3. The General Elephant in the Case $\operatorname{Bs}|-K_X| \neq \emptyset$

Let X be a canonical Gorenstein Fano threefold with $\operatorname{Bs}|-K_X| \neq \emptyset$. Choose a general elephant $\bar{S} \in |-K_X|$. By the Kawamata–Viehweg vanishing theorem $H^0(X, -K_X) \longrightarrow H^0(\bar{S}, -K_X|_{\bar{S}})$ is surjective, implying

$$\operatorname{Bs}|-K_X| = \operatorname{Bs}|-K_X|_{\bar{S}}| \neq \emptyset.$$

Let $\nu\colon S \to \bar{S}$ be a minimal desingularisation of $\bar{S}$. By Saint–Donat's results on linear systems on smooth K3 surfaces ([SD74] or [Shi89]),

$$\nu^*|-K_X|_{\bar{S}}| = |\Gamma + mf|,$$

where $m \geq 2$ and

i) $|f|$ is an elliptic pencil and
ii) $\Gamma = \operatorname{Bs}|\Gamma + mf| \simeq \mathbb{P}_1$ is a section.

Let $\Gamma' \subset S$ be an irreducible curve contracted by ν. Then $(\Gamma + mf) \cdot \Gamma' = 0$, implying $\Gamma \cap \Gamma' = \emptyset$ or $\Gamma = \Gamma'$. In the first case S and $\bar{S}$ are isomorphic near Γ and $\operatorname{Bs}|-K_X| \simeq \mathbb{P}_1 \subset \bar{S}_{\text{reg}}$. In the second case, Γ is contracted to a point, $\operatorname{Bs}|-K_X| = \{p\}$ and $p \in X_{\text{sing}}$. This is part of a result of Shin:

3.1. Theorem [Shin, [Shi89]**].** *Let X be a Gorenstein almost Fano threefold with at worst canonical singularities and assume* $\operatorname{Bs}|-K_X| \neq \emptyset$. *With $\bar{S} \in |-K_X|$ a general member we have*

i) *if* $\dim \operatorname{Bs}|-K_X| = 1$, *then scheme–theoretically* $\operatorname{Bs}|-K_X| \simeq \mathbb{P}_1$ *is contained in X_{reg} and* $\operatorname{Bs}|-K_X| \cap \operatorname{Sing}(\bar{S}) = \emptyset$*;*
ii) *if* $\dim \operatorname{Bs}|-K_X| = 0$ *then* $\operatorname{Bs}|-K_X|$ *consists of exactly one point and $\bar{S}$ has an ordinary double point at* $\operatorname{Bs}|-K_X|$. *In this case* $\operatorname{Bs}|-K_X| \subset \operatorname{Sing}(X)$.

Note that in the case $\operatorname{Bs}|-K_X| = \{p\}$ we have $(\Gamma + mf).\Gamma = 0$ on S, implying $m = 2$ and hence $(-K_X)^3 = 2$.

4. The Case $\dim \operatorname{Bs}|-K_X| = 0$

Let X be the complete intersection of a quadric Q in the linear variables and a sextic F_6 in $\mathbb{P}(1^4, 2, 3)$. If we choose F_6 general enough, then (see [Me99])

$$X \cap \{x_0 = x_1 = x_2 = x_3 = 0\} = [0:0:0:0:-1:1] = p$$

and X does not meet the singular locus of $\mathbb{P}(1^4, 2, 3)$. Then Q and F_6 are Cartier near X and by adjunction, $-K_X \simeq \mathcal{O}_{\mathbb{P}}(1)|_X$ and therefore $\operatorname{Bs}|-K_X| = \{p\}$. The rational map defined by $|-K_X|$ sends X to the quadric in $\mathbb{P}_3$ defined by Q.

4.1. Proposition. *If* $\dim \operatorname{Bs}|-K_X| = 0$*, then X is as above a complete intersection in $\mathbb{P}(1^4, 2, 3)$ of a quadric Q, defined in the first four linear variables, and a sextic F_6.*

Proof. (See [M82], [Me99], [IT01]). We know $(-K_X)^3 = 2$ (see the last section). By the Riemann–Roch theorem we get $h^0(-K_X) = 4$. Let

$$x_0, \ldots, x_3 \in H^0(-K_X)$$

be generating sections. We have $h^0(-2K_X) = 10 = \dim S^2H^0(-K_X)$. But $|-2K_X|$ is base point free, so there exists some

$$y \in H^0(-2K_X), \quad y \notin S^2H^0(-K_X).$$

Then we must have a nontrivial relation Q in $S^2H^0(-K_X)$. The x_i and y then define a 20–dimensional subspace of $H^0(-3K_X)$. By the theorem of Riemann–Roch $h^0(-3K_X) = 21$. Denote the missing function by $z \in H^0(-3K_X)$. Continuing in this way, we see that there must be a nontrivial relation F_6 in $H^0(-6K_X)$. In the end X is the complete intersection of Q and F_6 in $\mathbb{P}(1^4, 2, 3)$. $\square$

4.2. Remark. Since Q is singular at p, any $S \in |-K_X|$ is singular at p. If we choose Q and F_6 general, p will be a terminal point of X. If we take for Q the quadric cone, X will have canonical singularities along a curve.

5. The Examples for the Case $\dim \mathrm{Bs}\,|-K_X| = 1$

Let U be a canonical Gorenstein threefold. Assume that $|-K_U|$ contains a smooth K3 surface S such that

$$-K_U|_S = 2\Gamma_0 + mf$$

for some $m \geq 3$. Here $\mathbb{P}_1 \simeq \Gamma_0 \subset U_{\mathrm{reg}}$ and $|f|$ is an elliptic pencil as in section 3. Note that U is a hyperelliptic almost Fano threefold for $m \geq 4$.

Let $Y = Bl_{\Gamma_0}(U)$ be the blowup of U in Γ_0. The strict transform of S is a smooth K3 surface in $|-K_Y|$ which we denote by S as well. We have

$$-K_Y|_S = \Gamma_0 + mf,$$

implying $\mathrm{Bs}\,|-K_Y| = \Gamma_0 \simeq \mathbb{P}_1$. An anticanonical model X of Y is a canonical Gorenstein Fano threefold for which $\mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1$.

Examples for U as above are constructed as follows. For $m \geq 4$, U is almost Fano and the anticanonical map associated to $-K_U$ sends U to a variety of minimal degree

$$U \longrightarrow W \subset \mathbb{P}_{2m-2}.$$

Here S is sent to Σ_4, the fourth Hirzebruch surface. The idea is therefore to construct U as a ramified twosheeted covering of some variety of minimal degree, for which a general hyperplane section is isomorphic to Σ_4.

We now come to the examples in ii) in reverse order.

Examples ii), (c). The projective bundle

$$W = \mathbb{F}(m, m-4, 0), \quad m \geq 3$$

is a resolution of a cone over Σ_4. The projection of the underlying bundle onto the first two summands gives a split exact sequence and a smooth surface in $|\mathcal{O}_W(1)|$ isomorphic to Σ_4. For simplicity, we denote it by

$$\Sigma_4 \in |\mathcal{O}_W(1)|.$$

There exists a unique section $B \in |\mathcal{O}_W(1) - mF|$ meeting Σ_4 in its minimal section ξ_4. Below we prove that for $m \leq 12$ we may choose

$$D \in |\mathcal{O}_W(4) - (4m - 12)F|,$$

such that the square root of D yields a threefold U_m with at worst canonical singularities. We have

$$\mu\colon U_m \xrightarrow{2:1} \mathbb{F}(m, m-4, 0) \quad \text{and} \quad -K_{U_m} = \mu^*\mathcal{O}_W(1).$$

The section $\xi_4 = B \cap \Sigma_4 \subset D_{\text{reg}}$. Its reduced inverse image in U_m will be denoted by Γ_0. As in ii) (c) of Theorem 1.1, we denote by X_{2m-2} an anticanonical model of $Bl_{\Gamma_0}(U_m)$ for $3 \leq m \leq 12$. We claim that X_{2m-2} are canonical Gorenstein Fano threefolds with base locus $\mathrm{Bs}\,|-K_{X_{2m-2}}| \simeq \mathbb{P}_1$.

In order to prove this it suffices to show that for D general enough each U_m is a canonical Gorenstein threefold as in the beginning of this section. Since Σ_4 comes from a splitting sequence, $D \cap \Sigma_4$ is a general member of

$$|4\xi_4 + 12\mathfrak{f}|,$$

with $\mathfrak{f} \simeq \mathbb{P}_1$ a fiber of Σ_4. A general member of $|4\xi_4 + 12\mathfrak{f}|$ splits as $\xi_4 + C$ with $C \in |3\xi_4 + 12\mathfrak{f}|$ smooth and disjoint from ξ_4 (cf. section 2). The double covering of Σ_4 yields a smooth K3 surface $S \in |-K_{U_m}| = |\mu^*\mathcal{O}_W(1)|$ with

$$\mu_S\colon S \longrightarrow \Sigma_4$$

ramified along ξ_4 and C. The pullback of $\mathfrak{f}$ gives an elliptic pencil $|f|$ on S with the section Γ_0 lying over ξ_4 and $-K_{U_m}|_S = \mu_S^*\mathcal{O}(1) = 2\Gamma_0 + mf$. It remains to show that U_m has at worst canonical singularities for $3 \leq m \leq 12$ and $\Gamma_0 \subset U_{m,\text{reg}}$.

For $m = 3$ we can choose D and hence U_m smooth and there is nothing to prove. For $m \geq 4$, we always have

$$D = B + R$$

with $R \in |\mathcal{O}_W(3) - (3m - 12)F|$. Fiberwise $D \cap F$ consists of a line together with some cubic.

For $4 \leq m \leq 12$ we can take R to be irreducible, i.e., $D \cap F$ consists of a line and an irreducible cubic. For $m = 4$, the cubic is smooth, meeting the line transversally in three points. For $m \geq 5$, the line and the cubic intersect in one point, i.e., in a flex if the cubic is smooth. This gives an A–D–E singularity in the fiber, implying that U_m indeed has at worst canonical singularities for $3 \leq m \leq 12$. Since $R.\xi_4 = 0$ we can choose R disjoint from ξ_4. Hence $\Gamma_0 \subset U_{m,\text{reg}}$.

For $m \geq 13$ on the other hand, $R = R_1 + R_2 + R_3$ with $R_i \in |\mathcal{O}_W(1) - (m-4)F|$, so $D \cap F$ consists of four lines through a point. This means that over F we will not have Du Val singularities, implying that U_m is not canonical for $m \geq 13$.

5.1. Remark. The construction works for $m = 2$ as well. Here $\mathrm{Bs}\,|-K_{X_2}| = \{p\}$ and we get a special case of the threefold X in section 4 with Q the quadric cone (see Remark 4.2).

Example ii), (b). The product of S_1, a del Pezzo surface with canonical singularities of degree 1, and $\mathbb{P}_1$ is a classical example ([I80]). Choose 8 points on $\mathbb{P}_2$ general enough, such that the blowup $\hat{\mathbb{P}}_2$ of $\mathbb{P}_2$ in these points still has a nef anticanonical system, and denote by S_1 an anticanonical model of $\hat{\mathbb{P}}_2$. Then $|-K_{S_1}|$ is one dimensional by the Riemann–Roch theorem, its members corresponding to

elliptic curves passing through the eight points. These curves will meet in a ninth point, implying

$$\mathrm{Bs}\,|-K_{S_1}| = \{p\}.$$

Then the product $X = S_1 \times \mathbb{P}_1$ is a canonical Gorenstein Fano threefold with $\mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1$.

Example ii), (a). The blowup X in the intersection of two members of $|-\frac{1}{2}K_U|$ of the double cover U of the Veronese cone W, ramified along a cubic, is a classical example ([I80]). We give some details to show the connection to the above description.

The blowup of the Veronese cone in its vertex O yields

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(2)) \longrightarrow W.$$

The strict transform of a special hyperplane section through O gives a $\mathbb{P}_1$–bundle over a conic. It either decomposes into two copies of Σ_2 or gives one irreducible surface Σ_4.

The image of Σ_4 in W gives $\widehat{C}_4$, the cone over the rational normal curve of degree 4. In U, lying over $\widehat{C}_4$ we find a singular K3 surface $\bar{S} \in |-K_U|$ with a double point over O. In the reducible case, the two copies of Σ_2 induce $H_i \in |-\frac{1}{2}K_U|$ for $i = 1, 2$, and their intersection with $\bar{S}$ is the singular point.

In the blowup X of U along $H_1 \cap H_2$ the singularity of $\bar{S}$ is resolved, i.e., we get a smooth K3 surface $S \in |-K_X|$. The same formulas as above show

$$-K_X|_S = \Gamma + 2f$$

with Γ the -2–curve over the singularity and $|f|$ the induced elliptic pencil. If we choose H_1, H_2 general enough, then X will be a canonical Gorenstein Fano threefold with $\mathrm{Bs}\,|-K_X| \simeq \Gamma \simeq \mathbb{P}_1$.

6. The General Setting in the Case $\dim \mathrm{Bs}\,|-K_X| = 1$

(cf. [I80], [IP99]) By Shin's Theorem, $\Gamma = \mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1 \subset X_{\mathrm{reg}}$. We can write

(6.0.1)
$$N_{\Gamma/X} = \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(b), \quad a \geq b,$$

for some $a, b \in \mathbb{Z}$. A general elephant $\bar{S} \in |-K_X|$ may have double points, but $\Gamma \subset \bar{S}_{\mathrm{reg}}$. If $\nu\colon S \to \bar{S}$ denotes a resolution of the singular locus, then $\nu^*(-K_X) = \Gamma + mf$, $m \geq 3$, with $|f|$ an elliptic pencil and Γ a section (section 3). The numbers are related as follows:

$$-K_X \cdot \Gamma = m - 2 = a + b + 2.$$

Let $\sigma\colon X_\Gamma \to X$ be the blowup of X along Γ with exceptional divisor $E_\Gamma = \mathbb{P}(N^*_{\Gamma/X}) = \Sigma_{a-b}$. Then $|-K_{X_\Gamma}| = |\sigma^*(-K_X) - E_\Gamma|$ is free, defining a map onto some surface W ([R83]):

(6.0.2)
$$\begin{tikzcd} X_\Gamma \arrow[r, "\varphi"] \arrow[d, "\sigma"'] & W \subset \mathbb{P}_{m+1} \\ X \arrow[ur, dashed] & \end{tikzcd}$$

The surface W is of minimal degree, i.e.,

$$m = \deg(W) = \mathrm{codim}(W) + 1.$$

Again by del Pezzo's theorem, in our situation W is one of the following:

i) $\widehat{C}_m$, the cone over a rational normal curve of degree $m = a + b + 4 \geq 2$,
ii) Σ_{a-b}, $a \geq b$.

The map $E_\Gamma \to W$ is either an isomorphism or the contraction of the minimal section. The map $X_\Gamma \to W$ is (generically) an elliptic fibration, and since $-K_X$ is ample, any fiber over a point in W_{reg} is an irreducible, generically reduced curve of arithmetic genus one. We distinguish two cases.

The case W a smooth ruled surface. Here we denote by F_Γ the pullback to X_Γ of a fiber of W, and by $Z_{\Gamma,X}$ the pullback of the minimal section (or the second ruling in the case $W = \mathbb{P}_1 \times \mathbb{P}_1$). Note that $|F_\Gamma|$ descends to a pencil $|F|$ on X. Adjunction on E_Γ shows

$$-K_{X_\Gamma} = Z_{\Gamma,X} + (a+2)F_\Gamma.$$

Since $\Gamma \subset X_{\mathrm{reg}}$ and $Z_{\Gamma,X}$ meets E_Γ transversally near the minimal section ξ_{a-b} of E_Γ, $Z_{\Gamma,X}$ is smooth near $Z_{\Gamma,X} \cap E_\Gamma$, and $\sigma(Z_{\Gamma,X}) \simeq Z_{\Gamma,X}$ is smooth near Γ.

The case W a cone. Here we denote by F_Γ the strict transform in X_Γ of a line in W through the vertex O. Notice that this is just a Weil divisor. Let

$$h' \colon X'_\Gamma \longrightarrow X_\Gamma \tag{6.0.3}$$

be a $\mathbb{Q}$–factorialization of X_Γ with respect to F_Γ ([K88]). The map h' is small, X'_Γ is again Gorenstein with at worst canonical singularities, and the strict transform F'_Γ of F_Γ is $\mathbb{Q}$–Cartier. We can choose X'_Γ such that F'_Γ is h'–ample ([K88]). Since $\Gamma \subset X_{\mathrm{reg}}$, both X'_Γ and X_Γ are isomorphic near E_Γ. We denote the pullback of E_Γ to X'_Γ by E'_Γ. We claim (cf. [Ch99])

6.1. Lemma. *On X'_Γ, two general members of $|F'_\Gamma|$ do not intersect.*

Proof. Assume $F'_{\Gamma,1} \cap F'_{\Gamma,2} \neq \emptyset$. The intersection clearly is in the fiber over the vertex O of W. Choose an irreducible curve $C \subset F'_{\Gamma,1} \cap F'_{\Gamma,2}$. On the one hand, the restriction of some multiple of $F'_{\Gamma,2}$, which is Cartier, gives an effective Cartier divisor on $F'_{\Gamma,1}$ supported in the fiber over O, implying

$$F'_{\Gamma,2} \cdot C \leq 0.$$

On the other hand, since $F'_{\Gamma,1}$ and $F'_{\Gamma,2}$ do not meet on E'_Γ, we have $C \cap E'_\Gamma = \emptyset$. Since $-K_{X'_\Gamma} \cdot C = 0$ and $E'_\Gamma \cdot C = 0$ imply $h'^*\sigma^*(-K_X) \cdot C = 0$, the curve C must be h'–exceptional. Then, by our choice of X'_Γ,

$$F'_{\Gamma,2} \cdot C > 0.$$

Hence $F'_{\Gamma,1} \cap F'_{\Gamma,2} = \emptyset$. □

Denote by Y_Γ a terminal modification of X'_Γ. The pullback of F'_Γ to Y_Γ defines a pencil on Y_Γ, showing that the map to W factors over the blowup Σ_{a-b} of W in O. Near E'_Γ, Y_Γ and X'_Γ are isomorphic, and we can blow the divisor down to obtain Y, a terminal modification $h \colon Y \to X$ of X. We call the map $Y_\Gamma \to Y$ again σ and

end up with the following diagram:

(6.1.1)
$$\begin{tikzcd} & & \Sigma_{a-b} \arrow[d] \\ Y_\Gamma \arrow[r] \arrow[urr, "\psi"] \arrow[d, "\sigma"] & X_\Gamma \arrow[r] \arrow[d] & W \\ Y \arrow[r, "h"] & X & \end{tikzcd}$$

Below, we will study Y instead of X and think of X as an anticanonical model. Note that we have chosen Y as a terminal modification of a particular $\mathbb{Q}$–factorialization of X.

For simplicity, denote divisors on Y_Γ and X_Γ by the same letters: the exceptional divisor of $Y_\Gamma \to Y$ is again E_Γ, the curve $\mathrm{Bs}\,|-K_Y| = \Gamma$. The pullback of a general fiber of Σ_{a-b} to Y_Γ is F_Γ. By $Z_\Gamma + B_\Gamma$ we denote the pullback of the minimal section of Σ_{a-b} to Y_Γ, where Z_Γ denotes here the unique irreducible component that meets E_Γ in its minimal section, and B_Γ consists of the remaining components, disjoint from E_Γ. As above we get

$$-K_{Y_\Gamma} = Z_\Gamma + B_\Gamma + (a+2)F_\Gamma. \tag{6.1.2}$$

The pencil $|F_\Gamma|$ again descends to the pencil $|F|$ on Y. The surface Z_Γ is smooth near $E_\Gamma \cap Z_\Gamma$; we will denote the isomorphic images of Z_Γ and B_Γ in Y by Z and B.

6.2. Remark. The general member of the pencil $|F_\Gamma|$ is a smooth surface with a relatively minimal elliptic pencil. The intersection $F_\Gamma \cap (Z_\Gamma + B_\Gamma)$ is hence either smooth or one of Kodaira's exceptional fibers.

7. The Case W a Cone

7.1. Proposition. *If W is a cone, then $3 \le m \le 12$ and $X = X_{2m-2}$ is one of the threefolds constructed in section 5, Examples ii), (c). Here $W = \widehat{C}_m$.*

Proof. We use the notation from the last section. Since $-K_{X_\Gamma}$ is not ample on E_Γ, $b = -2$ and $a \ge 1$ in (6.0.1). We can hence use $a + b = m - 4$ to eliminate a and b and write everything in terms of m:

$$N_{\Gamma/X} = \mathcal{O}_{\mathbb{P}_1}(m-2) \oplus \mathcal{O}_{\mathbb{P}_1}(-2), \quad m \ge 3,$$

and $W = \widehat{C}_m$. In diagram (6.1.1), the map from Y_Γ to $\widehat{C}_m$ now factors over Σ_m.

We first assume that Z is h–nef and show that in this case Y is obtained by blowing up some Gorenstein threefold V along some smooth curve $\Gamma_0 \simeq \mathbb{P}_1 \subset V_{\mathrm{reg}}$, such that Z is the exceptional divisor. We compute

$$Z \cdot \Gamma = -2 \quad \text{and} \quad -K_Y \cdot \Gamma = m - 2 > 0.$$

Hence $[\Gamma]$ is contained in the K_Y–negative part of $\overline{NE}(Y)$. This part is polyhedral, spanned by K_Y–negative extremal rays. The divisor Z is negative on $[\Gamma]$ and nonnegative on any K_Y–trivial curve by assumption. We conclude that Z must be negative on at least one extremal ray. Let

$$\phi\colon Y \longrightarrow V \tag{7.1.1}$$

be the contraction of this ray. By [Ben85], the contraction is divisorial, contracting Z either to a curve or to a point. We claim

7.2. Lemma. *The map $\phi\colon Y \to V$ in (7.1.1) is the blowup of a smooth rational curve $\Gamma_0 \subset V_{\mathrm{reg}}$ with normal bundle*

$$N_{\Gamma_0/V} = \mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(m-4).$$

The contraction is in direction of $|F|$. There exists a smooth K3 surface $S \in |-K_V|$, such that

$$-K_V|_S = 2\Gamma_0 + mf$$

with $|f|$ an elliptic pencil induced by $|F|$, and $\Gamma_0 \simeq \mathbb{P}_1$ a smooth section.

7.3. Remark. The threefold V is a hyperelliptic Gorenstein almost Fano threefold of degree $(-K_V)^3 = 4m-8$ for $m \geq 4$. For $m = 3$, the anticanonical system is nef on any curve $\neq \Gamma_0$, while

$$-K_V \cdot \Gamma_0 = m - 4 = -1.$$

For the case $m = 3$ (as well as $m = 2$) see also [DPS93].

Proof of Lemma 7.2 and Remark 7.3. Since Z_Γ meets E_Γ transversally in the minimal section, we have $\Gamma \subset Z_{\mathrm{reg}}$. We compute

$$\deg N_{\Gamma/Z} = Z_\Gamma \cdot_{Y_\Gamma} E_\Gamma^2 = m - 2 > 0. \tag{7.3.1}$$

Let us first show $Z \not\simeq \mathbb{P}_1 \times \mathbb{P}_1$. If $Z \simeq \mathbb{P}_1 \times \mathbb{P}_1$, then $B \neq 0$, implying that B meets Z in some curve. By (7.3.1) Γ is ample on Z. Then $\Gamma \cap B \neq \emptyset$, which is impossible since B maps to X_{sing}, while $\Gamma \subset X_{\mathrm{reg}}$.

If Z is mapped to a point, then by [Cu88], $Z \simeq \mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$ or the quadric cone. Since Z comes with a pencil and $Z \not\simeq \mathbb{P}_1 \times \mathbb{P}_1$, all these cases are impossible. By [Cu88], $Y = Bl_{\Gamma_0}(V)$ the blowup of V in some curve $\Gamma_0 \subset V_{\mathrm{reg}}$, which is locally a complete intersection. From $\deg N_{\Gamma/Z} = m - 2 > 0$ we conclude that Γ maps surjectively onto Γ_0, and from $\Gamma \subset Z_{\mathrm{reg}}$ we infer that Γ_0 must be smooth. Then

$$Z = \mathbb{P}(N^*_{\Gamma_0/V}) \simeq \Sigma_e \quad \text{for some } e > 0,$$

where $e > 0$ follows from $Z \not\simeq \mathbb{P}_1 \times \mathbb{P}_1$. It is now clear that ϕ is in direction of $|F|$, i.e., fiberwise ϕ contracts a -1–curve in F. Denote the induced pencil on V by $|F_V|$. Notice that $Z \simeq \Sigma_e$ implies $B \neq 0$.

1.) Any curve in $Z_\Gamma \cap B_\Gamma$ is contracted by $Y_\Gamma \to X_\Gamma$, and therefore B intersects Z set theoretically in the minimal section ξ_e of $Z = \Sigma_e$. Since Γ does not meet ξ_e, we conclude $\Gamma = \xi_e + (m-2)\mathfrak{f}_e$, where $\mathfrak{f}_e$ is a fiber of Σ_e. From $\Gamma \cdot_Z \Gamma = m-2$ (7.3.1) we infer $e = m-2$. Moreover, $-K_Y \cdot \xi_e = 0$ implies

$$N_{\Gamma_0/V} = \mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(m-4).$$

By the adjunction formula, $-K_V \cdot \Gamma_0 = m-4$, hence $(-K_V)^3 = 4m-8$.

2.) Let $S \in |-K_Y|$ be general. Since S meets Z transversally in Γ, its image in V is a special member of $|-K_V|$. Identifying S with its image in V we find

$$-K_V|_S = 2\Gamma_0 + mf,$$

where $|f|$ is an elliptic pencil and Γ_0 is a section (see section 5). If $C \subset V$ is an irreducible curve such that $-K_V \cdot C < 0$, then $S \cdot C < 0$ and $C \subset S$. Then $-K_V \cdot C = (2\Gamma_0 + mf) \cdot C < 0$ so that $\Gamma_0 \cdot C < 0$ and hence $C = \Gamma_0$, $m = 3$. $\square$

The argument before Lemma 7.2 showing the contractibility of Z in Y requires Z being h–nef. In order to achieve this we might have to change the terminal modification by running the relative $(K_Y+\epsilon Z)$–program, $\epsilon \in \mathbb{Q}^+$, $\epsilon \ll 1$, with respect to $h\colon Y \to X$.

The contraction of any $(K_Y+\epsilon Z)$–negative extremal ray in $\overline{NE}(Y/X)$ is small; the curves contracted are K_Y–trivial and contained in Z. After finitely many flops, we end up with the following picture:

$$
\begin{tikzcd}
Y \arrow[rr, dashed, "\chi"] \arrow[dr, "h"'] & & Y^+ \arrow[dl, "h^+"] \\
& X &
\end{tikzcd}
\tag{7.3.2}
$$

([KM98], Theorem 6.14 and Corollary 6.19). Here Y^+ is again a terminal Gorenstein threefold with $-K_{Y^+}$ big and nef, having X as an anticanonical model. The map χ is rational and an isomorphism in codimension one. We superscribe any strict transform under χ with a "+"–sign. Since $K_{Y^+}+\epsilon Z^+$ is h^+–nef, Z^+ is h^+–nef. As above we conclude that Z^+ is contractible in Y^+.

Lemma 7.2 holds for Y^+ instead of Y as long as $|F^+|$ is still spanned on Y^+. This need not be the case. Recall that we have chosen Y as a terminal modification of some $\mathbb{Q}$–factorialization X' of X; in the above program we might flop some horizontal curves in Z, thereby producing a base locus.

7.4. Lemma. *$|F^+|$ is spanned unless $m=3$ and $(Z^+,\mathcal{O}_{Z^+}(Z^+))=(\mathbb{P}_2,\mathcal{O}_{\mathbb{P}_2}(-2))$.*

Here $|F^+|$ restricted to Z^+ corresponds to lines through a given point.

Proof. Assume that $|F^+|$ is not spanned. Let

$$\phi^+\colon Y^+ \longrightarrow V^+$$

be the divisorial contraction as in (7.1.1), contracting Z^+. In order to decide what Z^+ is, we again use the classification from [Cu88]. If Z^+ maps to a curve and $\mathfrak{f}$ denotes the general fiber, then $Z^+\cdot\mathfrak{f}=-1$ and $-K_{Y^+}\cdot\mathfrak{f}=1$. On Y^+ we have

$$-K_{Y^+}=Z^++B^++mF^+. \tag{7.4.1}$$

Since $\mathrm{Bs}\,|F^+|\cap Z^+\neq\emptyset$ we must have $F^+\cdot\mathfrak{f}>0$. From $B^+\cdot\mathfrak{f}\geq 0$ we conclude $0<m\mathfrak{f}\cdot F^+\leq 2$ which is impossible since $m>2$.

If Z^+ goes to a point, then $(Z^+,\mathcal{O}_{Z^+}(Z^+))$ is either $(\mathbb{P}_2,\mathcal{O}(-1))$ or $(\mathbb{P}_2,\mathcal{O}(-2))$ or $(Q_2\subset\mathbb{P}_3,\mathcal{O}(-1))$. Near Γ the two surfaces Z and Z^+ are isomorphic. With the original pencil on Z we conclude that Z^+ contains a smooth rational curve that meets another irreducible curve in a single point. From $Z^+\cdot\Gamma=-2$ we infer $(Z^+,\mathcal{O}_{Z^+}(Z^+))=(\mathbb{P}_2,\mathcal{O}(-2))$. Then $|F^+|$ restricted to Z^+ is a family of lines. Using $-K_{Y^+}\cdot\Gamma=m-2$ and the adjunction formula, we find $m=3$. The proof of the Lemma is complete. □

Lemma 7.2 also holds in the exceptional case $(Z^+,\mathcal{O}_{Z^+}(Z^+))=(\mathbb{P}_2,\mathcal{O}_{\mathbb{P}_2}(-2))$ for some terminal modification of X', we only cannot argue as above. Instead, we proceed as follows.

We first run the relative $(K_Y+\epsilon Z)$–program with respect to $Y\to X'$, where X' is the above $\mathbb{Q}$–factorialization of X. In the end we may assume that Z is at least nef on every K_Y–trivial curve contained in a fiber of the pencil $Z\to\mathbb{P}_1$. Omitting

some details, we conclude that a single flop of a K_Y–trivial section of Z transforms Y into Y^+ in (7.3.2) and Z into $Z^+ = \mathbb{P}_2$ as above. Then

$$Z \simeq \Sigma_1$$

and $Z \cdot \mathfrak{f} = -1$ for the general fiber $\mathfrak{f} \simeq \mathbb{P}_1$. We conclude that Z must be negative on at least one extremal ray in $\overline{NE}(Y/\mathbb{P}_1)$ and conclude Lemma 7.2 as above.

For the proof of Proposition 7.1 it remains to show that V in Lemma 7.2 is a terminal modification of U_m in section 5. In order to prove this, we consider the system

$$|-K_V + \lambda F_V|, \quad \lambda \geq 0,$$

and choose λ such that $m + \lambda \geq 4$. Restricted to S we get $2\Gamma_0 + (m+\lambda)f$, which is now big and nef. Then $-K_V + \lambda F_V$ is big and nef and by the Kawamata–Viehweg vanishing theorem $H^1(\mathcal{O}_V(\lambda F_V)) = H^1(\mathcal{O}_V(K_V + (-K_V + \lambda F_V)) = 0$ implying surjectivity of

$$H^0(V, \mathcal{O}_V(-K_V + \lambda F_V)) \longrightarrow H^0(S, \mathcal{O}_S(2\Gamma_0 + (m+\lambda)f)).$$

Then, since $|F_V|$ is free and $|2\Gamma_0 + (m+\lambda)f|$ is free, $|-K_V + \lambda F_V|$ is free. For $\lambda \geq 1$ and $m + \lambda \geq 5$, any irreducible curve having zero intersection with $-K_V + \lambda F_V$ must lie in a member of $|F_V|$. This follows immediately from $-K_V + \lambda F_V = (-K_V + (\lambda - 1)F_V) + F_V = \text{nef} + \text{nef}$. The system is free, for example, if we choose $\lambda = 1$, for $m \geq 4$, and $\lambda = 2$, for $m = 3$.

Fix this choice from now on. The map associated to $|-K_V + \lambda F_V|$ is generically 2–to–1 sending V to a variety of minimal degree

$$\nu \colon V \longrightarrow W \subset \mathbb{P}_{2m+3\lambda-2}.$$

Since W comes with a pencil $|F_W|$, it must be a scroll. We may rescale the entries such that $-K_V \simeq \nu^* \mathcal{O}_W(1)$. Then $W \simeq \mathbb{F}(d_1, d_2, d_3)$, $d_1 \geq d_2 \geq d_3 \geq -1$, where $d_3 = -1$ in the case $m = 3$, while $d_3 \geq 0$ for $m \geq 4$. Stein factorization of $V \to W$ leads to a canonical Gorenstein threefold U and a double cover

$$\mu \colon U \longrightarrow W \simeq \mathbb{F}(d_1, d_2, d_3),$$

such that $-K_U = \mu^* \mathcal{O}_W(1)$. Hence μ is ramified along a reduced divisor

$$D \in |\mathcal{O}_W(4) - 2(d_1 + d_2 + d_3 - 2)F_W|.$$

From $(\mathcal{O}_W(1))^3 = \frac{1}{2}(-K_V)^3 = 2m - 4$ we infer

$$d_1 + d_2 + d_3 = 2m - 4.$$

The only section of $H^0(V, -K_V - mF_V)$ is the one corresponding to the image of B in V (cf. (6.1.2)). Since μ is fiberwise ramified along a quartic, we also have $h^0(W, \mathcal{O}_W(1) - mF_W) = 1$, implying

$$d_1 = m, \quad d_2 < m.$$

In the special case $m = 3$ we have $d_3 = -1$ and $W \simeq \mathbb{F}(3, 0, -1)$. It remains to consider the case $m \geq 4$.

Denote the image of B in W by B_W. If $d_3 > 0$, then $2B_W$ is a component of D. But D is reduced, hence we must have $d_3 = 0$. Then $d_1 = m$, $d_2 = m - 4$, i.e.,

$$V \longrightarrow U \longrightarrow W \simeq \mathbb{F}(m, m-4, 0).$$

We have seen in section 5, that $U = U_m$ can never have canonical singularities for $m \geq 13$, hence $m \leq 12$.

Back on the surface $S \in |-K_V|$ in Lemma 7.2, we see that S is generically a double cover of some member $H \in |\mathcal{O}_W(1)|$. The map ν sends S to $\mathbb{F}(m, m-4)$ and Γ_0 lies over the minimal section, which is the restriction of the above divisor B_W. In particular, Γ_0 is not contracted by $V \to U_m$ and does not meet any curve contracted, i.e., $\Gamma_0 \subset U_{m,\mathrm{reg}}$ and V is isomorphic to U_m near Γ_0. This completes the proof of Proposition 7.1. □

8. The Case W a Ruled Surface

This case is as in [I80]. Instead of Y and Y_Γ we focus on X and X_Γ, and diagram (6.0.2). We use the notation introduced in section 6.

8.1. Proposition. *In the case $W \simeq \Sigma_{a-b}$, $a > b$, X is the blowup of a sextic in $\mathbb{P}(1^3, 2, 3)$ along an irreducible curve of arithmetic genus one (and $a = 0$, $b = -1$, $m = 3$).*

Proof. Since $-K_{X_\Gamma}$ is ample on E_Γ, we have $b \geq -1$ and $a \geq 0$. Hence

$$Z_{\Gamma,X} \cdot \xi_{a-b} = b - a < 0 \quad \text{and} \quad -K_{X_\Gamma} \cdot \xi_{a-b} = b + 2 > 0,$$

where $\xi_{a-b} = E_\Gamma \cap Z_{\Gamma,X}$ is the minimal section of E_Γ. Since $Z_{\Gamma,X}$ is trivial on any K_{X_Γ}–trivial curve, we conclude that $Z_{\Gamma,X}$ must be negative on at least one extremal ray in $\{K_{X_\Gamma} < 0\}$. Denote by

$$\phi_X \colon X_\Gamma \longrightarrow V_X$$

the contraction of this ray. It is a birational map with exceptional set $Z_{\Gamma,X}$ by [Ben85]. Since $Z_{\Gamma,X}$ contains K_{X_Γ}–trivial curves, it is contracted to a curve.

If $Z_{\Gamma,X}$ is singular along a curve, then its normalization is a smooth ruled surface. The second map implies that it is $\mathbb{P}_1 \times \mathbb{P}_1$. Since $\xi_{a-b} \subset Z_{\Gamma,X,\mathrm{reg}}$ does not meet the singular locus, we must have

$$\deg N_{\xi_{a-b}/Z_{\Gamma,X}} = a = 0,$$

implying $b = -1$. If $Z_{\Gamma,X}$ is smooth in codimension one, then $h^1(Z_{\Gamma,X}, \mathcal{O}_{Z_{\Gamma,X}}) \leq 1$ by [R83] and Iskovskikh's original argument applies: using the ideal sequence of $Z_{\Gamma,X}$ and the identity $-K_{X_\Gamma} = Z_{\Gamma,X} + (a+2)F_\Gamma$ (cf. section 6), we see

$$h^1(Z_{\Gamma,X}, \mathcal{O}_{Z_{\Gamma,X}}) = h^2(X_\Gamma, \mathcal{O}_{X_\Gamma}(-Z_{\Gamma,X})) = h^1(X_\Gamma, \mathcal{O}_{X_\Gamma}(-(a+2)F_\Gamma)).$$

Then the ideal sequence of $(a+2)$ general members of $|F_\Gamma|$

$$0 \longrightarrow \mathcal{O}_{X_\Gamma}(-(a+2)F_\Gamma) \longrightarrow \mathcal{O}_{X_\Gamma} \longrightarrow \mathcal{O}_{(a+2)F_\Gamma} \longrightarrow 0$$

yields $h^0(\mathcal{O}_{(a+2)F_\Gamma}) - 1 \leq 1$, hence $a \leq 0$.

Since $E_\Gamma \cdot \xi_{a-b} = a = 0$, the image Z_X of $Z_{\Gamma,X}$ is still contractible. We can even explicitly give the supporting divisor: denote the image of F_Γ in X by F. They are Cartier, since $\Gamma \subset X_{\mathrm{reg}}$. The supporting divisor is

$$H = Z_X + F \in \mathrm{Pic}(X),$$

which is big and nef. Indeed, $\sigma^* H = Z_{\Gamma,X} + F_\Gamma + E_\Gamma$. Since $Z_{\Gamma,X} + F_\Gamma = \varphi^*(\xi_1 + \mathfrak{f})$ is nef, and $\sigma^* H$ restricted to E_Γ is trivial, H is nef. A direct computation shows $H^3 = 1$. By the Base Point Free Theorem, $|kH|$ is free for $k \gg 0$, defining a birational contraction

$$\phi \colon X \longrightarrow V,$$

contracting Z_X to a curve. The base locus $\Gamma \subset Z_X$ is contracted to a point, the general fiber of the elliptic pencil on Z_X is a section. The variety V is again a Gorenstein Fano threefold with canonical singularities and

$$K_X = \phi^* K_V + Z_X.$$

From $\phi^* K_V = K_X - Z_X = -2H$ we conclude that $-K_V$ is divisible by 2 in $\mathrm{Pic}(V)$. From $H^0(X, kH) = 1 + \frac{k}{6}(8 + 3k + k^2)$ we see that V is a sextic in $\mathbb{P}(1^3, 2, 3)$. $\square$

8.2. Proposition. *If $W \simeq \mathbb{P}_1 \times \mathbb{P}_1$, then $X \simeq \mathbb{P}_1 \times S_1$, where S_1 denotes a normal del Pezzo surface of degree* 1 *(and $a = b = 0$, $m = 4$).*

Proof. In this case, $Z_{\Gamma,X}$ is the pullback of one ruling of $W = \mathbb{P}_1 \times \mathbb{P}_1$. The general fiber of $Z_{\Gamma,X}$ is a smooth elliptic curve, and $Z_{\Gamma,X}$ meets the singular locus of X_Γ at most in points. Going from X_Γ to Y_Γ, we see

$$a \leq 0.$$

Since $E_\Gamma \simeq W$, we have $a = b$, and X Fano implies $a = b = 0$. Since φ followed by the natural projection $W \to \mathbb{P}_1$ contracts all the fibers of $\sigma \colon X_\Gamma \to X$ to points, we obtain an induced map

$$X \longrightarrow \mathbb{P}_1$$

with general fiber $F = \sigma(F_\Gamma)$ and section Γ, where F is a normal del Pezzo surface of degree one. We have $-K_{X_\Gamma} = Z_{\Gamma,X} + 2F_\Gamma$. As above,

$$-K_X = Z_X + 2F,$$

and we see that Z_X is nef, so $|kZ_X|$ is free for $k \gg 0$. The map defined by $|kZ_X|$ is a $\mathbb{P}_1$–bundle with section F and fiber Γ. As in [I80] we conclude that $X \simeq F \times \mathbb{P}_1$ is a product. $\square$

References

[Ben85] X. Benveniste, *Sur le cone des* 1*–cycles effectifs en dimension* 3, Math. Ann. **272**, 257–265 (1985).

[Ber07] E. Bertini, *Introduzione alla geometria proiettiva degli iperspazi*, Enrico Spoerri, Pisa, 1907.

[Ch99] I.A. Chel'tsov, *Three-Dimensional Fano Varieties of Integer Index*, Math. Notes **66**, 360–365 (1999).

[Cu88] S. Cutkosky, *Elementary Contractions of Gorenstein Threefolds*, Math. Ann. **280**, 521–525 (1988).

[dP85] P. del Pezzo, *Sulle superficie di ordine n immerse nello spazio di n+1 dimensioni*, Nap. rend. XXIV, 212–216 (1885).

[DPS93] J.-P. Demailly, T. Peternell, M. Schneider, *Kähler manifolds with numerically effective Ricci class*, Comp. Math. **89**, 217–240 (1993).

[IT01] S. Ishii, M. Tomari, *Hypersurface non-rational singularities which look canonical from their Newton boundaries*, Math. Z. **237**, 125–147 (2001).

[I80] V.A. Iskovskikh, *Anticanonical models of three-dimensional algebraic varieties*, J. Soviet Math. **13**, 745–814 (1980).

[IP99] V.A. Iskovskikh, Yu.G. Prokhorov, *Algebraic Geometry V: Fano varieties*, Springer 1999.

[K88] Y. Kawamata, *Crepant blowing-up of* 3*-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. Math. **127**, 93–163 (1988).

[KM98] J. Kollár, S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge Univ. Press 1998.

[Me99] M. Mella, *Existence of good divisors on Mukai varieties*, J. Alg. Geom. **8**, 197–206 (1999).

[M82] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math. **116**, 133–176 (1982).
[M88] S. Mori, *Lectures on 3-dimensional algebraic varieties*, Utah University, Fall 1988.
[R79] M. Reid, *Canonical 3-folds*, Algebraic Geometry Angers, 273–310 (1979).
[R83] M. Reid, *Projective morphisms according to Kawamata*, Preprint (1983).
[SD74] B. Saint-Donat, *Projective models of K3 surfaces*, Am. J. Math. **96**, 602–639 (1974).
[Shi89] K.-H. Shin, 3-*dimensional Fano Varieties with Canonical Singularities*, Tokyo J. Math. **12**, 375–385 (1989).
[Sho80] V.V. Shokurov, *Smoothness of a general anticanonical divisor on a Fano variety*, Math. USSR, Izv. **14**, 395–405 (1980).

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